Exact solutions of the Schrödinger equation in D-dimensions for the pseudoharmonic potential plus ring-shaped potential

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Abstract

We present analytically the exact energy bound-states solutions of the Schrödinger equation in D-dimensions for a pseudoharmonic potential plus ring-shaped potential of the form $V(r,\theta) = D_e \left(\frac{r}{r_e} - \frac{r_e}{r} \right)^2 + \frac{\beta \cos^2 \theta}{r^2 \sin^2 \theta}$ by means of the conventional Nikiforov-Uvarov method. We also give a clear recipe of how to obtain an explicit solution to the radial and angular parts of the wave functions in terms of orthogonal polynomials. The total energy of the system is different from the pseudoharmonic potential because of the contribution of the angular part. The general results obtained in this work can be reduced to the standard forms given in the literature.

Keywords: Energy eigenvalues and eigenfunctions, pseudoharmonic potential, ring-shaped potential, non-central potentials, Nikiforov and Uvarov method.

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I. INTRODUCTION

The solution of the fundamental dynamical equations is an interesting phenomenon in many fields of physics and chemistry. To obtain the exact ℓ-state solutions of the Schrödinger equation (SE) are possible only for a few potentials and hence approximation methods are used to obtain their solutions [1]. According to the Schrödinger formulation of quantum mechanics, a total wave function provides implicitly all relevant information about the behaviour of a physical system. Hence if it is exactly solvable for a given potential, the wave function can describe such a system completely. Until now, many efforts have been made to solve the stationary SE with anharmonic potentials in two dimensions ($2D$) and three dimensions ($3D$) [2-6] with many applications to molecular and chemical physics. The study of the SE with these potentials provides us with insight into the physical problem under consideration. However, the study of SE with some of these potentials in the arbitrary dimensions D is presented in (cf. Ref. [7] and the references therein). Furthermore, the study of the bound state processes is also fundamental to understanding molecular spectrum of a diatomic molecule in quantum mechanics [8].

The Harmonic oscillator [9,10] and H-atom (Coulombic) [9-11] problems have been thoroughly studied in D-dimensional space quantum mechanics for any angular momentum ℓ. These two problems are related together and hence the resulting second-order differential equation has the normalized orthogonal polynomial function solution.

On the other hand, the pseudoharmonic potential may be used for the energy spectrum of linear and non-linear systems [12,13]. It is generally used for discussion of molecular vibrations. Additionally, this potential possesses advantages over the harmonic and leads to equally spaced energy levels. One of the advantages of the pseudoharmonic potential over the harmonic oscillator is that it can be treated exactly in three as well as in one dimension. Besides, the Pseudoharmonic and Mie-type potentials [12-14] are two exactly solvable potentials other than the Coulombic and anharmonic oscillator [9-11].

An exactly complete bound-state solutions of the $3D$ SE with pseudoharmonic potential,

anharmonic oscillator-like potential with inclusion of a centrifugal potential barrier does not complicate the solutions, which are available in closed form. Recently, the solution was also carried out by using orthogonal polynomial solution method and also by performing a proper transformation procedures [15]. Besides, the analytical solutions of the D-dimensional radial SE with some diatomic molecular potentials like pseudoharmonic [15] and modified Morse or Kratzer-Fues [16] potential are also solved by selecting a suitable ansatz to the wave function [17].

Chen and Dong [18] found a new ring-shaped (non-central) potential and obtained the exact solution of the SE for the Coulomb potential plus this new ring-shaped potential which has possible applications to ring-shaped organic molecules like cyclic polyenes and benzene. The complete exact energy bound-state solution and the corresponding wave functions of a class of non-central potentials [19] have been solved recently by means of the conventional Nikiforov-Uvarov (NU) method [19-25].

Recently, Cheng and Dai [26], proposed a new potential consisting from the modified Kratzer's potential [27] plus the new proposed ring-shaped potential in [18]. They have presented the energy eigenvalues for this proposed exactly-solvable non-central potential in 3D-SE throughout the NU method. The two quantum systems solved by Refs [18,26] are closely relevant to each other as they deal with a Coulombic field interaction except for inclusion of a centrifugal potential barrier acts as a repulsive core which is for any arbitrary angular momentum ℓ prevents collapse of the system in any dimensional space due to this additional perturbation to the original angular momentum barrier. In a very recent works [28,29], we have given a clear recipe of how to obtain analytically the exact energy eigenvalues and the corresponding normalized wave functions of the Schrödinger equation in D-dimensions with the proposed modified Kratzer plus ring-shaped potential [28] and the modified Coulomb plus ring-shaped potential [29] by means of the conventional NU method [19-29].

The purpose of the paper is to solve the SE for the pseudoharmonic potential plus this new ring-shaped potential offered in [18]. This new proposed potential falls among in the

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class of non-central potential which have already been solved in our previous work [19]. In spherical coordinates, we have given this appellation to the non-central potential

$$V(r,\theta) = D_e \left(\frac{r}{r_e} - \frac{r_e}{r} \right)^2 + \beta \frac{ctg^2\theta}{r^2} = V_1(r) + \frac{V_2(\theta)}{r^2}, \tag{1}$$

where $D_e = kr_0^2/8$ is the dissociation energy between two atoms in a solid, r_e is the equilibrium intermolecular seperation and β is positive real constant. The potential (1) reduces to the pseudoharmonic potential in the limiting case of $\beta = 0$ [13,15]. The NU method [19-29] has been used to solve the SE for this new potential (1).

This work is organized as follows: in section II, we shall briefly introduce the basic concepts of the NU method. Section III is mainly devoted to the exact solution of the Schrödinger equation in D-dimensions for this quantum system by means of the NU method. Finally, the relevant results are discussed in section IV.

II. BASIC CONCEPTS OF THE METHOD

The NU method is based on reducing the second-order differential equation to a generalized equation of hypergeometric type [20]. In this sense, the Schrödinger equation, after employing an appropriate coordinate transformation $s = s(r)$, transforms to the following form:

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)} \psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \psi_n(s) = 0, \tag{2}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second-degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. Using a wave function, $\psi_n(s)$, of the simple ansatz:

$$\psi_n(s) = \phi_n(s)y_n(s), \tag{3}$$

reduces (2) into an equation of a hypergeometric type

$$\sigma(s)y_n''(s) + \tau(s)y_n'(s) + \lambda y_n(s) = 0, \tag{4}$$

where

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$$\sigma(s) = \pi(s)\frac{\phi(s)}{\phi'(s)}, \tag{5}$$

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \ \tau'(s) < 0, \tag{6}$$

and λ is a parameter defined as

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \qquad n = 0, 1, 2, ... \tag{7}$$

The polynomial $\tau(s)$ with the parameter s and prime factors show the differentials at first degree be negative. It is worthwhile to note that λ or λ_n are obtained from a particular solution of the form $y(s) = y_n(s)$ which is a polynomial of degree n. Further, the other part $y_n(s)$ of the wave function (3) is the hypergeometric-type function whose polynomial solutions are given by Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right], \tag{8}$$

where B_n is the normalization constant and the weight function $\rho(s)$ must satisfy the condition [20]

$$\frac{d}{ds}w(s) = \frac{\tau(s)}{\sigma(s)}w(s), \ w(s) = \sigma(s)\rho(s). \tag{9}$$

The function π and the parameter λ are defined as

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)}, \tag{10}$$

$$\lambda = k + \pi'(s). \tag{11}$$

In principle, since $\pi(s)$ has to be a polynomial of degree at most one, the expression under the square root sign in (10) can be arranged to be the square of a polynomial of first degree [20]. This is possible only if its discriminant is zero. In this case, an equation for k is obtained. After solving this equation, the obtained values of k are substituted in (10). In addition, by comparing equations (7) and (11), we obtain the energy eigenvalues.

III. EXACT SOLUTIONS OF THE QUANTUM SYSTEM WITH THE NU METHOD

A. Seperating variables of the Schrödinger equation

The proposed potential (1) can be simply rewritten in the form of isotropic harmonic oscillator plus inverse quadratic plus ring-shaped potential as

$$V(r,\theta) = ar^2 + \frac{b}{r^2} + \beta\frac{\cos^2\theta}{r^2\sin^2\theta} + c, \ \beta > 0 \tag{12}$$

where $a = D_e r_e^{-2}$, $b = D_e r_e^2$ and $c = -2D_e$.

The potential in (12) solved for the limiting case of $\beta = 0$ by using the orthogonal polynomial solution method [15] and by the Rydberg-Klein-Rees (RKR) procedures [13]. In fact the energy spectrum for the potential in (12) can be obtained directly by considering it as one case of the general non-central seperable potentials discussed previously in [19].

Our aim is to derive analytically the exact energy spectrum for a moving particle in the presence of a potential (12) in a very simple way. The D-dimensional space SE, in spherical polar coordinates, for the potential (12) is [28-30]

$$-\frac{\hbar^2}{2\mu}\left\{\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right.\right.$$

$$\left.\left. + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} - \frac{2\mu\beta}{\hbar^2}\frac{\cos^2\theta}{\sin^2\theta}\right] + \frac{2\mu}{\hbar^2}\left(E - ar^2 - \frac{b}{r^2} - c\right)\right\}\psi(r,\theta,\varphi) = 0, \tag{13}$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ being the reduced mass of the two particles and $\psi(r,\theta,\varphi)$ being the total wave function separated as follows

$$\psi_{n\ell m}(r,\theta,\varphi) = R(r)Y_\ell^m(\theta,\varphi), \ R(r) = r^{-(D-1)/2}g(r), \ Y_\ell^m(\theta,\varphi) = H(\theta)\Phi(\varphi). \tag{14}$$

On substituting equation (14) into (13) leads to a set of second-order differential equations:

$$\frac{d^2\Phi(\varphi)}{d\varphi^2} + m^2\Phi(\varphi) = 0, \tag{15}$$

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \frac{m^2}{\sin^2\theta} - \frac{2\mu\beta}{\hbar^2}\frac{\cos^2\theta}{\sin^2\theta} + \ell(\ell+D-2)\right]H(\theta) = 0, \tag{16}$$

$$\left[\frac{1}{r^{D-1}}\frac{d}{dr}\left(r^{D-1}\frac{d}{dr}\right) - \frac{\ell(\ell+D-2)}{r^2}\right]R(r) + \frac{2\mu}{\hbar^2}\left[E - ar^2 - \frac{b}{r^2} - c\right]R(r) = 0. \tag{17}$$

The solution in (15) is periodic and must satisfy the period boundary condition $\Phi(\varphi+2\pi) = \Phi(\varphi)$ from which we obtain [9,10]

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}}\exp(\pm im\varphi), \quad m = 0, 1, 2, ... \tag{18}$$

Further, equation (16) representing the angular wave equation takes the simple form

$$\frac{d^2H(\theta)}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dH(\theta)}{d\theta} + \left[\ell(\ell+D-2) - \frac{m^2 + (2\mu\beta/\hbar^2)\cos^2\theta}{\sin^2\theta}\right]H(\theta) = 0, \tag{19}$$

which will be solved in the following subsection.

B. The solutions of the angular equation

In order to apply NU method, we introduce a new variable $s = \cos\theta$, Eq. (19) is then rearranged as the universal associated-Legendre differential equation [26,28,29]

$$\frac{d^2H(s)}{ds^2} - \frac{2s}{1-s^2}\frac{dH(s)}{ds} + \frac{\nu'(1-s^2) - m'^2}{\sin^2\theta}H(\theta) = 0, \tag{20}$$

where

$$\nu' = \ell'(\ell'+D-2) = \ell(\ell+D-2) + 2\mu\beta/\hbar^2 \quad \text{and} \quad m' = \sqrt{m^2 + 2\mu\beta/\hbar^2}. \tag{21}$$

The solution of this equation has already been solved by the NU method in [26,28,29]. However, the aim in this subsection is to solve with different parameters resulting from the D-space-dimensions of Schrödinger equation. Upon letting $D = 3$, we can readily obtain the standard case given in [26]. By comparing Eqs. (20) and (2), the corresponding polynomials are obtained. In terms of the variable s, we have

$$\tilde{\tau}(s) = -2s, \quad \sigma(s) = 1 - s^2, \quad \tilde{\sigma}(s) = -\nu's^2 + \nu' - m'^2. \tag{22}$$

Inserting the above expressions into equation (10) and taking $\sigma'(s) = -2s$, one obtains the following function:

$$\pi(s) = \pm\sqrt{(\nu' - k)s^2 + k - \nu' + m'^2}. \tag{23}$$

Following the method, the polynomial $\pi(s)$ is found to have the following four possible values:

$$\pi(s) = \begin{cases} m's & \text{for } k_1 = \nu' - m'^2, \\ -m's & \text{for } k_1 = \nu' - m'^2, \\ m' & \text{for } k_2 = \nu', \\ -m' & \text{for } k_2 = \nu'. \end{cases} \tag{24}$$

Imposing the condition $\tau'(s) < 0$, for equation (6), one selects the following physically valid solutions:

$$k_1 = \nu' - m'^2 \quad \text{and} \quad \pi(s) = -m's, \tag{25}$$

which yields form equation (6)

$$\tau(s) = -2(1 + m')s. \tag{26}$$

Using Eqs (7) and (11), the following expressions for λ are obtained, respectively,

$$\lambda = \lambda_n = 2n(1 + m') + n(n - 1), \tag{27}$$

$$\lambda = \nu' - m'(1 + m'). \tag{28}$$

We compare Eqs (27) and (28) and from the definition $\nu' = \ell'(\ell' + D - 2)$, the new angular momentum ℓ' values are obtained as

$$\ell' = -\frac{(D - 2)}{2} + \frac{1}{2}\sqrt{(D - 2)^2 + 4(n + m')(n + m' + 1)}, \tag{29}$$

which can be easily reduced to the simple form

$$\ell' = n + m', \tag{30}$$

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in $3D$ [26]. Using Eqs (3)-(5) and (8)-(9), the wave function can be written as,

$$H_{m'}(\theta) = \sqrt{\frac{(2\ell'+1)(\ell'-m')!}{2(\ell'+m')!}} \sin^{m'}(\theta) P_n^{(m',m')}(\cos\theta), \tag{31}$$

where

$$n = -\frac{(1+2m')}{2} + \frac{1}{2}\sqrt{(2\ell'+1)^2 + 4\ell'(D-3)}, \tag{32}$$

with ℓ' is given explicitly in (29).

Hence, we are left to solve Eq. (17). It transforms, after lengthy but straightforward algebra, into the following simple form [30]:

$$\frac{d^2g(r)}{dr^2} + \left[\frac{2\mu}{\hbar^2}(E-c) - \frac{2\mu a}{\hbar^2}r^2 - \frac{\tilde{\nu} + (2\mu b/\hbar^2)}{r^2}\right]g(r) = 0, \tag{33}$$

where

$$\tilde{\nu} = \frac{1}{4}(M-1)(M-3), \ M = D + 2\ell. \tag{34}$$

Obviously, the two particles in Eq. (33) interacting via anharmonic oscillator potential plus inclusion of a centrifugal potential acts as a repulsive core which for any arbitrary ℓ prevents collapse of the system in any space dimension due to this additional centrifugal potential barrier. Hence, Eq. (33) has to be solved by using the NU method in the next subsection.

C. The solutions of the radial equation

The aim of this subsection is to solve the problem with a different radial separation function $g(r)$ in any arbitrary dimensions. We now study the bound-states (real) solution $E > c$ of Eq. (33). By employing the suitable transformation, $s = r^2$, and letting

$$\varepsilon = \sqrt{\frac{2\mu}{\hbar^2}(E-c)}, \ \alpha = \frac{2\mu a}{\hbar^2}, \ \gamma = \tilde{\nu} + \frac{2\mu b}{\hbar^2}, \tag{35}$$

one can transforms Eq. (33) into the following form:

$$\frac{d^2g(s)}{ds^2} + \frac{1}{2s}\frac{dg(s)}{ds} + \frac{1}{(2s)^2}\left[\varepsilon^2 s - \alpha s^2 - \gamma\right]g(s) = 0. \tag{36}$$

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To apply the conventional NU-method, Eq. (36) is compared with (2), the corresponding polynomials are obtained

$$\tilde{\tau}(s) = 1, \quad \sigma(s) = 2s, \quad \tilde{\sigma}(s) = \varepsilon^2 s - \alpha s^2 - \gamma. \tag{37}$$

The polynomial $\pi(s)$ in Eq. (10) can be found by substituting Eq. (37) and taking $\sigma'(s) = 2$. Hence, the polynomial $\pi(s)$ is

$$\pi(s) = \frac{1}{2} \pm \frac{1}{2}\sqrt{4\alpha s^2 + 4(2k - \varepsilon^2)s + 4\gamma + 1}. \tag{38}$$

According to this conventional method, the form of the expression under the square root in Eq. (38) is to be set equal to zero and solved for the two roots of k which can be readily obtained as

$$k = \frac{\varepsilon^2}{2} \pm \frac{1}{2}\sqrt{\alpha(4\gamma + 1)}. \tag{39}$$

In view of that, we arrive at the following four possible functions of $\pi(s)$:

$$\pi(s) = \begin{cases} \frac{1}{2} + \frac{1}{2}\left[2\sqrt{\alpha}s + \sqrt{4\gamma + 1}\right] & \text{for } k_1 = \frac{\varepsilon^2}{2} + \frac{1}{2}\sqrt{\alpha(4\gamma + 1)}, \\ \frac{1}{2} - \frac{1}{2}\left[2\sqrt{\alpha}s + \sqrt{4\gamma + 1}\right] & \text{for } k_1 = \frac{\varepsilon^2}{2} + \frac{1}{2}\sqrt{\alpha(4\gamma + 1)}, \\ \frac{1}{2} + \frac{1}{2}\left[2\sqrt{\alpha}s - \sqrt{4\gamma + 1}\right] & \text{for } k_2 = \frac{\varepsilon^2}{2} - \frac{1}{2}\sqrt{\alpha(4\gamma + 1)}, \\ \frac{1}{2} - \frac{1}{2}\left[2\sqrt{\alpha}s - \sqrt{4\gamma + 1}\right] & \text{for } k_2 = \frac{\varepsilon^2}{2} - \frac{1}{2}\sqrt{\alpha(4\gamma + 1)}. \end{cases} \tag{40}$$

The correct value of $\pi(s)$ is chosen such that the function $\tau(s)$ given by Eq. (6) will have negative derivative [20]. So we can select the physically valid solutions to be

$$k = \frac{\varepsilon^2}{2} - \frac{1}{2}\sqrt{\alpha(4\gamma + 1)} \quad \text{and} \quad \pi(s) = \frac{1 + \sqrt{4\gamma + 1}}{2} - \sqrt{\alpha}s, \tag{41}$$

which give

$$\tau(s) = 2 + \sqrt{4\gamma + 1} - 2\sqrt{\alpha}s, \ \tau'(s) = -2\sqrt{\alpha} < 0. \tag{42}$$

Using Eqs (7) and (11), the following expressions for λ are obtained, respectively,

$$\lambda = \lambda_n = 2N\sqrt{\alpha}, \ N = 0, 1, 2, ..., \tag{43}$$

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$$\lambda = \frac{\varepsilon^2}{2} - \frac{1}{2}\sqrt{\alpha(4\gamma + 1)} - \sqrt{\alpha}. \tag{44}$$

Hence, the energy eigenvalues are:

$$E_N = c + \sqrt{\frac{\hbar^2 a}{2\mu}}\left(4N + 2 + \sqrt{(M-1)(M-3) + 8\mu b/\hbar^2 + 1}\right). \tag{45}$$

Substituting

$$(M-1)(M-3) = 4\tilde{\nu} = (D-2)^2 + 4\ell'(\ell' + D - 2) - 8\mu\beta/\hbar^2 - 1, \tag{46}$$

with ℓ' defined in (29), Eq. (45) transforms into the form

$$E_N = c + \sqrt{\frac{\hbar^2 a}{2\mu}}\left(4N + 2 + \sqrt{(D-2)^2 + 4\ell'(\ell' + D - 2) + 8\mu(b-\beta)/\hbar^2}\right), \tag{47}$$

where $N, \ell' = 0, 1, 2, ...$ Equation (47) contains the contributions coming from the angular-dependent part of the SE for the pseudoharmonic potential plus ring-shaped potential as well.

(i) For $3D$-pseudoharmonic potential plus ring-shaped potential case, we use transformation of parameters $a = D_e r_e^{-2}$, $b = D_e r_e^2$ and $c = -2D_e$ in Eq. (47):

$$E_N = -2D_e + \sqrt{\frac{2\hbar^2 D_e}{\mu r_e^2}}\left(2N + 1 + \sqrt{(n + m' + 1/2)^2 + 2\mu(D_e r_e^2 - \beta)/\hbar^2}\right), \tag{48}$$

where $N, n, m' = 0, 1, 2, ..$ and m' is defined in Eq. (21).

(ii) For $3D$-pseudoharmonic potential case, we use the transformation of the parameters $a = D_e r_e^{-2}$, $b = D_e r_e^2$, $c = -2D_e$ in Eq. (47):

$$E_N = -2D_e + \sqrt{\frac{2\hbar^2}{\mu r_e^2}}\left(2N + 1 + \sqrt{\left(\ell + \frac{1}{2}\right)^2 + 2\mu D_e r_e^2/\hbar^2}\right), \quad N, \ell = 0, 1, 2, ... \tag{49}$$

where $\ell = n + m$ and it is found to be consistent with Ref. [15].

In what follows, we attempt to find the radial wavefunctions for this potential. Using $\tau(s)$, $\pi(s)$ and $\sigma(s)$ in Eqs (5) and (9), we find the first part of the wave function

$$\phi(s) = \exp\left(-\frac{\sqrt{\alpha}}{2}s\right)s^{(1+\sqrt{4\gamma+1})/4}, \tag{50}$$

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and the weight function

$$\rho(s) = \exp\left(-\sqrt{\alpha}s\right) s^{\sqrt{4\gamma+1}/2} \tag{51}$$

which is useful for finding the second part of the wave function. Besides, we substitute Eq. (51) into the Rodrigues relation in Eq. (8) and obtain one of the wave functions in the form

$$y_n(s) = B_n \exp\left(\sqrt{\alpha}s\right) s^{-\sqrt{4\gamma+1}/2} \frac{d^N}{ds^N} \left(s^{N+\sqrt{4\gamma+1}/2} \exp\left(-\sqrt{\alpha}s\right)\right), \tag{52}$$

where B_n is a normalization constant. Hence, the wave function $g(s)$ can be written in the form of the generalized Laguerre polynomials as

$$g(r) = C_{N,L} \exp\left(-\frac{\sqrt{\alpha}}{2}s\right) s^{(1+\sqrt{4\gamma+1})/4} L_N^{(L+\frac{1}{2})}(\sqrt{\alpha}s), \tag{53}$$

where

$$L = \frac{1}{2}\left[\sqrt{(D-2)^2 + 4\ell'(\ell' + D - 2) + 8\mu(b-\beta)/\hbar^2} - 1\right]. \tag{54}$$

Finally, the radial wave functions of the Schrödinger equation are obtained

$$R(r) = C_{N,L} r^{L-(D-3)/2} \exp\left(-\frac{\sqrt{\alpha}}{2}r^2\right) L_N^{(L+\frac{1}{2})}(\sqrt{\alpha}r^2), \tag{55}$$

where $C_{N,L}$ is the normalization constant to be determined below. Using the normalization condition, $\int_0^\infty R^2(r)r^{D-1}dr = 1$, and the orthogonality relation of the generalized Laguerre polynomials, $\int_0^\infty z^{\eta+1}e^{-z}\left[L_n^\eta(z)\right]^2 dz = \frac{(2n+\eta+1)(n+\eta)!}{n!}$, we have

$$C_{N,L} = \sqrt{\frac{2\left(\sqrt{2\mu D_e}/\hbar r_e\right)^{L+3/2} N!}{\Gamma(L+N+3/2)}}. \tag{56}$$

Therefore, we may express the normalized total wave functions as

$$\psi(r,\theta,\varphi) = \sqrt{\frac{\left(\sqrt{2\mu D_e}/\hbar r_e\right)^{L+3/2} (2\ell'+1)(\ell'-m')!N!}{\pi(\ell'+m')!\Gamma(N+L+3/2)}} r^{L-(D-3)/2}$$

$$\times \exp\left(-\sqrt{\frac{\mu D_e}{2\hbar^2 r_e^2}}r^2\right) L_N^{(L+1/2)}\left(\sqrt{\frac{2\mu D_e}{\hbar^2 r_e^2}}r^2\right) \sin^{m'}(\theta) P_n^{(m',m')}(\cos\theta) \exp(\pm im\varphi). \tag{57}$$

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On the other hand, the wave functions in (57) can be reduced to their standard forms in [15] as one sets $\beta = 0$. Therefore, we finally have

$$\psi(r, \theta, \varphi) = \sqrt{\frac{\left(\sqrt{2\mu D_e}/\hbar r_e\right)^{L+3/2} (2\ell+1)(\ell-m)!N!}{\pi(\ell+m)!(N+L+1/2)!}} r^{L-(D-3)/2}$$

$$\times \exp\left(-\sqrt{\frac{\mu D_e}{2\hbar^2 r_e^2}} r^2\right) L_N^{(L+1/2)}\left(\sqrt{\frac{2\mu D_e}{\hbar^2 r_e^2}} r^2\right) \sin^m(\theta) P_n^{(m,m)}(\cos\theta) \exp(\pm im\varphi). \qquad (58)$$

where

$$L + \frac{1}{2} = \sqrt{(\ell+1/2)^2 + 2\mu D_e r_e^2/\hbar^2}, \ \ell = n + m. \qquad (59)$$

By choosing appropriate values of the parameters D_e and r_e, the desired exact energy spectrum and the corresponding wave functions can be calculated for every case.

IV. CONCLUSIONS

We have easily obtained the exact bound state solutions of the D dimensional radial SE for a diatomic molecule with the pseudoharmonic potential plus ring-shaped potential by means of the conventional NU method for any $N\ell'm'$. The NU method can be applied systematically to both radial and angular parts of the wave function. Further, the wave functions are expressed in terms of special orthogonal functions such as Laguerre and Jacobi polynomials [31]. The presented procedure in this study is systematical and efficient in finding the exact energy spectra and corresponding wave functions of the Schrödinger equation for various diatomic molecules. This new proposed potential can be reduced to the standard pseudoharmonic potential in the $3D$, which appears to describe the molecular vibrations quite well, by doing the following transformations: $m' \rightarrow m$, $\ell' \rightarrow \ell = n + m$. We point out that this method is simple and promising in producing the exact bound state solution for energy states and wave functions for further anharmonic oscillator-type potential plus new ring-shaped potential.

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